JGP-Vol. 2, n. 1, 1985

Relativistic quantum mechanics with non conserved number of particles

PHILIPPE DROZ-VINCENT
Chaire de Physique Mathématique
Collège de France
11, place Marcelin Berthelot
75231 Parix Cedex 05 (France)

Abstract. *A covariant scheme is given for the second quantization of directly interacting particles.*
In a model, the space of interacting state vectors is explicitly constructed, at the price of an invariant alteration of the original framework, which is necessary owing to the presence of divergences.

1. INTRODUCTION

In the last decade considerable interest has been devoted to relativistic dynamics of directly interacting particles [1].

This point of view, rather than being in conflict with field theory must be understood as complementary.

Of course, the various formulations of relativistic particle dynamics are not especially tailored for the description of local interactions. Most probably, these new developments of particle dynamics provide a quite general framework in which local interactions can take place, but as a very particular case. Reformulating in this framework the well-known realistic interactions (e.g. electromagnetic),

Key-Words: Relativistic Action-At-A-Distance.
1980 Mathematics Subject Classification: *81 E 99, 81 G 25, 81 M 05.*

which are of local nature, is therefore a widely open question that we should keep in mind [2].

Nevertheless, our understanding of N-body relativistic dynamic has undergone substantial progresses in the recent years.

If we consider the matter seriously, it is logical to go one step further, taking into account creation and annihilation of particles.

This leads naturally to a relativistic second quantization which is not necessarily a theory of fields, though it may eventually be exciting to make a contact with conventional Q.F.T.

In this spirit, we have proposed a scheme [3] which is the logical continuation of the so called *a priori hamiltonian formalism* of Predictive Mechanics.

Another method inspired by similar motivations has been independently suggested by Coester and Polyzou [4].

Our approach can be summarized as follows:

The object of our study is not a field but a system of particles, the number of them being unspecified and in general not constant.

Reference to the customary lagrangian canonical formalism is radically dropped and replaced by a many-hamiltonian formalism directly inspired from N-body covariant mechanics.

As a redult our description is basically off-shell and an infinite sequence of *hamiltonian generators* is supposed to determine the dynamics. At the price of a linear rearrangement, these operators form an infinite set of *squared-mass operators*.

Since we consider stable particles, states of physical relevance are on the mass shell. The mass-shell space is selected by diagonalization of the squared-mass operators. (a generalized eigenvalue problem [5]). An evolution operator U can be formed, which allows for an infinitely-many-time-dependent formalism where the scattering problem should be posed.

This conceptual framework is simple, but mathematical difficulties are tremendous, and we cannot be too ambitious from the start. Therefore

a) In contrast with the authors of Ref. [4] we shall not intend here to implement cluster separability.

b) We do not yet assume indiscernability.

As shown in Ref. [3], this picture can be displayed explicitly and with complete mathematical rigor in the case of free particles, where it turns out to be equivalent to the conventional treatment by Q.F.T. except naturally that the selection of positive energy states is not automatically implied by the eigenvalue equations *unless* suitable restriction is made, from the outset, about the big space in which the whole description is imbedded.

The first true problem is to modify the hamiltonian gerators of the free case in order to obtain a non trivial model.

Especially, the conservation of the number of particles should be broken in a way which respects Poincaré invariance.

This requirement strongly suggests that the basic ingredients for this modification are annihilation/creation operators associated with a suitable invariant (but off-shell!) one-particle state.

Indeed they change the number of particles and we control their transformation properties under the Poincaré group (covariance).

Going further in this direction leads to define an off-shell free field operator which is not to be used in any Lagrangian formalism, but should serve as a stone in the construction of additional terms in our scheme [6].

But, in order to avoid unnecessary technical complications, in the present work particles will be treated as distinguishable.

Thus, instead of considering creation/annihilation, or generalized free field [7] operators, we shall deal with *contraction* and *tensorial-product* operators [8], which are less popular but geometrically equivalent and, in the absence of symetrization, more easy to manipulate.

With the help of these tools we aim at producing a model which must be non trivial but as simple and tractable as possible. In other words, we look for a prototype which could throw light on the theory before we resume a more accurate description.

Among a lot of various simplifications we neglect spin and consider only the case of massive particles.

This paper is reorganized as follows:

Section 2 countains mathematical preliminaries and a survey of the conventions we use.

Section 3 is devoted to the main lines of our second quantization scheme.

In section 4 a tentative model is sketched and its difficulties are analyzed.

Section 5 displays a general method which alterates the original formalism and replaces the divergences by meaningful expressions.

In section 6 we indicate how to solve the mass-shell conditions which characterize the space of physical states.

2. NOTATIONS, CONVENTIONS

Space-time signature is $+ - - -$.

We take $c = \hbar = 1$.

Greek labels α, $\beta = 0, 1, 2, 3$ are ommitted whenever possible, the scalar product of four-vectors being denoted by a dot, $\ell \cdot x = \ell_\alpha x^\alpha$, etc. No summation over particle indices, unless explicitly specified. For each integer value of n we introduce the indices

$$a_n = 1, 2, \ldots, n.$$

Let $\partial_{n,a}$ be the differentiation operator $\partial/\partial x_a^\alpha = \partial_{a\alpha}$ when it is necessary to specify that it acts in the n-particle space. Ommitting the dummy Greek label we write

$$\Box_{n,a_n} = \partial_{n,a_n} \cdot \partial_{n,a_n}.$$

For instance $\Box_{2,1}$ is the dalembertian operator relative to the variable x_1 in the space of two-boby wave functions.

$L^2(\mathbb{R}^4)$ is the space of square integrable functions on space-time.

The integrals we write are understood from $-\infty$ to $+\infty$.

The state vectors we consider are of the form

$$\Phi = (\varphi_0, \ldots, \varphi_n(x_1, x_2 \ldots x_n), \ldots). \tag{2.1}$$

Various spaces are defined by imposing condizions on the φ_n. It might be convenient to imbed them in a «big space» E which countains all the vectors we need. We may define this big space by demanding that every φ_n is a tempered distribution in all its arguments.

Thus, in contrast to more conventional habits, a state like Φ in equ. (2.1) is generally neither on the mass shell nor in the Hilbert space $H = \overset{\oplus}{\Sigma} L^2(\mathbb{R}^{4n})$.

However reference to H is precious as giving a reliable domain of validity for some statements.

The scalar product in $L^2(\mathbb{R}^4)$ is noted $\langle \varphi, \psi \rangle$ and the same notation is extended to the scalar product $\langle \Phi, \Psi \rangle$ in H without risk of confusion. This notation may be used also when Φ is a regular state and Ψ a generalized state in the sense of rigged Hilbert space theory [5].

The particle number operator N and Poincaré transformations are defined in E by the usual formulas

$$N \Phi = (0, \varphi_1, \ldots, n\, \varphi_n, \ldots,) \tag{2.2}$$

$$\Lambda \Phi = (\varphi_0, \Lambda \varphi_1, \ldots, \Lambda \varphi_n, \ldots,) \tag{2.3}$$

and for instance

$$P_\mu \Phi = (0, \partial_1 \varphi_1, (\partial_1 + \partial_2)\, \varphi_2, \ldots,). \tag{2.4}$$

The no-particle state $(1, 0, 0, \ldots)$ is considered as the vacuum for free particles (mathematical vacuum). It is Poincaré invariant and annihilated by P_μ, but not the only one in E to share these properties.

Of fundamental importance are the contraction operator $C[\varphi]$ and the tensorial product operator $T[\varphi]$. We only shall consider the case where φ is the constant function $\varphi(x) \equiv 1$ and set $C[1] = C$, $T[1] = T$ Definition formulas are the following:

Let $E^{(n)}$ be the n-particle space. C is first defined as a map: $E^{(n)} \to E^{(n-1)}$ by

$$C\varphi_n = \int \varphi_n(y, x_1, \ldots, x_{n-1})\, \mathrm{d}^4 y. \tag{2.5}$$

Then C is extended to E (Using the same notation without risk of confusion)

$$C\Phi = (C\varphi_1, C\varphi_2, \ldots, C\varphi_{n+1} \ldots) \tag{2.6}$$

T is first a map $E^{(n)} \to E^{(n+1)}$

$$T\varphi_n = \varphi_n(x_2, x_3 \ldots x_{n+1}) \tag{2.7}$$

$$T\varphi_0 = \varphi_0 \qquad \text{as a constant function of } x. \tag{2.8}$$

Then T is extended to E

$$T\Phi = (0, T\varphi_0, T\varphi_1 \ldots T\varphi_{n-1} \ldots). \tag{2.9}$$

While T is always defined, the domain of C must be characterized by the requirement that the integral (2.5) converges in some well-defined sense, which amounts to restrict the n-particle space by a condition $\varphi_n \in \mathscr{T}^{(n)} = \mathscr{T}^{\otimes n}$ where $\mathscr{T}$ is a suitable space in which $\int \varphi(x)\, \mathrm{d}^4 x$ always converges. In view of our purpose, and for the sake of simplicity, let us assume that $\mathscr{T}$ is the space of *functions* which are globally integrable from $-\infty$ to $+\infty$, in a large enough sense [9]. Note also that for $\ell \neq 0$ the plane waves are in $\mathscr{T}$, according to

$$\int e^{i\ell \cdot x}\, \mathrm{d}^4 x = 0. \tag{2.10}$$

In contrast, the constant function $\varphi(x) \equiv 1$ is not in $\mathscr{T}$, and therefore T, as defined in (2.7) always leads out of $\mathscr{T}^{(n+1)}$.

Call $\overset{\oplus}{\Sigma}\mathscr{T}^{(n)}$ (symbolically) the subspace of E resulting from the assignment that $\varphi_n \in \mathscr{T}^{(n)}$. We see that T, defined as in (2.9) always leads out of $\overset{\oplus}{\Sigma}\mathscr{T}^{(n)}$.

For simplicity again, we shall not display the separation of positive and negative energies. Still, let us point out that it would be possible to restrict E by demanding that the Fourier transformed of each φ_n has support limited by the future light cone, with respect to each one of its arguments. This defining E^+.

Using $C[1]$ and $T[1]$ as we do would be compatible with such a procedure, as the Fourier transformed of $\varphi(x) \equiv 1$ is concentrated on the vertex, which belongs to both parts of the light cone.

3. SECOND QUANTIZATION

A. Free Particles

The infinitely-many-hamiltonian formulation applies trivially to free particles.

In this case, the conventional description requires that the state vector satisfies the condition

$$(\Box_{a_n} + m^2)\, \psi_n(x_1, \ldots, x_n) = 0 \tag{3.1}$$

for $a_n = 1, 2, \ldots, n$.

By a slight abuse of language all the states which fulfill (3.1) are usually said to be on the mass shell, though their component ψ_0 is not in general solution of an equation like (3.1). In particular the vacuum does not correspond to a fixed value of m.

We shall respect the traditional denomination, but it is useful to introduce the concept of *exclusive mass shell* characterized by the additional assumption of no component on the vacuum, that is

$$\psi_0 = 0. \tag{3.2}$$

Now the mass shell space K_m is the direct sum of the exclusive mass-shell space K_m^{ex} with the one-dimensional no-particle space.

Let us introduce the hamiltonian generators by the formula

$$h_{n,a_n} \Phi = \left(0, \ldots, 0, -\frac{1}{2} \Box_{a_n} \varphi_n, 0 \ldots\right) \tag{3.3}$$

and the squared-mass operators by

$$M_{a_1 a_2} \ldots \Phi = \left(0, -\frac{1}{2} \Box_{a_1} \varphi_1, \ldots, -\frac{1}{2} \Box_{a_n} \varphi_n, \ldots\right) \tag{3.4}$$

which means the linear rearrangement

$$M_{a_1 a_2} \ldots = h_1 + h_{2,a_2} + \ldots h_{n,a_n} + \ldots . \tag{3.5}$$

It is clear that the genralized eigengalue equations

$$M_{a_1 a_2 a_3} \ldots \Psi = \frac{1}{2} m^2 \Psi \tag{3.6}$$

are equivalent with $\Psi \in K_m^{ex}$. Reconstruction of the total space K_m is achieved by dirict sum with the vacuum space (with is a *null* eigenspace of the M-operators. Thus, for $m \neq 0$ the bare vacuum space is actually orthogonal to K_m^{ex}). Note that the need to distinguish the h from the M is a genuine feature of the second quantization in its most general many-time formalism. This technical detail corresponds to the postulate [10] that the evolution operator has the form

$$U = U_1(U_{2,1} U_{2,2})(U_{3,1} U_{3,2} U_{3,3}) \ldots \tag{3.7}$$

with

$$U_{n,a_n} = \exp i\, \tau_{n,a_n} h_{n,a_n}. \tag{3.8}$$

(No summation).

All the h-operators commute among themselves and there is no problem of mathematical rigor in (3.5) or (3.7). The constants of the motion being the operators which commute with the hamiltonian generators, the conservation of linear or angular momentum is derived without invoking the lagrangian canonical formalism.

B. Interacting Particles

We postulate that interacting particles are described by suitable hamiltonian generators which are no longer given by (3.3). We retain the rearrangement formula (3.5) but (3.3) and (3.4) are modified in order to take the interaction into account.

The new operators h_{n,a_n} must keep along with the following requirements

a) Commutation among temselves

$$[h_{n,a_n}, h_{p,a_p}] = 0.$$

b) They remain functionally independent.

c) Commutation with the generators of Poincaré algebra.

d) Hermiticity.

The evolution operator remains formally defined by (3.7).

Note that (a) legitimates a lot of formal calculations dealing with U in scattering theory. Moreover (a) is the logical generalization of the predictivity conditions we know in N-boby dynamics [11].

The *interacting* exclusive *mass shell* is characterized by eq. (3.6) where now $M_{a_1 a_2 \dots}$ are the modified squared-mass operators. It is worthwile noticing that (a) and (3.5) imply that the M operators are constants of the motion, and the equations (3.6) form a compatible system.

After selection of the exclusive mass-shell space it will be relevant to consider eventually the extension of it by a suitable physical vacuum which might be distinct from the bare no-partical state [12].

This point will be discussed later, in the context of more specific assumptions.

The problem is to satisfy the requirements a) b) c) d) in a nontrivial way. The less we can demand is that

I) $[h_{n,a_n}, N]$ not all vanish in order to ensure that N is no longer a constant of the motion.

II) The new hamiltonian generators do not all commute with the free ones, in order to have

$$[U, U_{\text{free}}] \neq 0.$$

Otherwize the scattering properties are trivial.

Haag's theorem [13] is not *a priori* an objection, since our approach is not founded on the canonical lagrangian formalism and the usual hamiltonian *density* is not employed. However, the consistency of (I - II) with (c) requires special care and is postponed to the next section.

Let us indicate a straightforward method for constructing operators which satisfy all the above requirements except *perhaps* (c). From now on, *quantities labelled with a bar refer to the free particle system*. For example $\overline{U} = U_{\text{free}}$ etc..

Let B be an invertible operator which satisfies

(I′) $\qquad [B, N] \neq 0$

(II′) $\qquad [B, \overline{h}]$ not all vanish, thus $[B, \overline{U}] \neq 0$.

It is clear that setting

$$h_{n,a_n} = B\,\overline{h}_{n,a_n} B^{-1} \tag{3.9}$$

automatically we fulfill (a) (b).

The squared mass operators are given by

$$M_{a_1 a_2 \dots} = B\,\overline{M}_{a_1 a_2 \dots} B^{-1} \tag{3.10}$$

and we have also $U = B\overline{U}B^{-1}$ and eq. (3.5) (3.7) (3.8) hold without mathematical difficulties.

The exclusive mass shell K^{ex} is in one-to-one correspondance with $\overline{K}^{ex}$, and its elements given by

(3.11) $$\Phi = B\overline{\Phi}$$

which solves (3.6) if $\overline{\Phi} \in \overline{K}^{ex}$.

The complete mass shell space is obtained by addition of the *physical vacuum* Φ_0, which we assume to be of the form

(3.12) $$\Phi_0 = B\,\overline{\Phi}_0$$

where $\overline{\Phi}_0$ is the no-free-particle state $(1, 0, \dots)$. Note that Φ_0 necessarily a zero-mass state.

Thus finally (3.11) defines all the interacting states, if $\overline{\Phi}$ is an arbitrary free-particle state on the mass shell (vacuum included). When, moreover, B is *unitary*, then (d) also is satisfied and U is unitary like $\overline{U}$.

Such a model is still *spectrally* trivial in this sense that the eigenstates of M are in one-to-one correspondance with those of $\overline{M}$.

Nevertheless the particle number is not constant and the scattering problem is far from trivial, because $[U, \overline{U}]$ does not vanish.

The crucial question is whether B can be chosen in such a way that condition (c) also is satisfied. It would be sufficient that B commutes with the generators of the Poincaré algebra. If so, we could derive the conservation of linear and angular momenta, and Φ_0 would be an invariant no-momentum state.

4. THE NAIVE MODEL

In order to have B unitary it is natural to choose

(4.1) $$B = \exp i\,k\,A$$

where k is a coupling constant and A hermitian.

One is left with the problem of finding A which does not commute with N and such that $[A, \overline{h}_{n,a_n}]$ not all vanish.

From a general view point it is natural to attempt the construction of A from an *invariant* combination (including derivatives and integrations) of *covariant* generalized free fields which act off shell [6] [7].

Still we way fear that no such invariant A exists at all. It seems that a simple solution is

(4.2) $$A = C + T$$

with C and T defined in eqs. (2.6) and (2.9) respectively. Indeed C and T commute with Poincaré transformations and break the conservation of N (For instance computing $(C + T)\,\overline{\Phi}_0$ we obtain a one-particle state). That A does not commute with all the $\overline{h}$ is easy to check. Compute for instance the action of $[C + T, \overline{h}_{3,2}]$ on a 2-body state $(0, 0, \varphi_2(x_1 x_2), 0 \ldots)$. The only non vanishing contribution comes from $\overline{h}_{3,2} T$.

Recalling that $C = C[1]$ and $T = T[1]$ one expects that $C = T^+$ and therefore A is hermitian [14]. The truth is a little bit more subtle, all we can prove being

$$\langle \Phi, C\,\Psi \rangle = \langle T\,\Phi, \Psi \rangle \tag{4.3}$$

where Ψ is restricted by the condition that ψ_n are test functions (in Schwartz space). In fact $T\Phi$ cannot be normalized (for $\Phi \neq 0$) and (4.3) makes sense in a rigged Hilbert space.

The most naive treatment would ignore mathematical difficulties in a first order perturbation expansion according to the formulas

$$B = 1 + i\,k\,A \tag{4.4}$$

$$\Phi = \overline{\Phi} + i\,k\,A\,\overline{\Phi} \tag{4.5}$$

and, omitting various particle indices

$$h = \overline{h} + i\,k\,[A, \overline{h}] \tag{4.6}$$

$$M = \overline{M} + i\,k\,[A, \overline{M}]. \tag{4.7}$$

Explicit computation of (4.5) is very easy when $\overline{\Phi}$ is an n-body plane wave. For instance when $n = 1$ one finds

$$\Phi = (0, e^{i\varrho\cdot x}, i\,k\,e^{i\varrho\cdot x_2}, 0 \ldots). \tag{4.8}$$

The vacuum of interacting particles is at first order

$$\Phi_0 = (1, i\,k, 0, \ldots). \tag{4.9}$$

Unfortunately any effort to go beyond the first order is plagued by divergences. The origin of this trouble is that the constant function $\varphi(x) = 1$ is not in $L^2(\mathbb{R}^4)$ which implies that $T[1]$ is not an operator in H.

Actually T always leads out of the domain where C is defined, hence the product CT diverges like the volume of space-time.

Finally A^2 and the higher powers of A diverge. It is clear that this drawback is the price paid for the use of the constant function which guarantees Poincaré invariance. Of course C and T can be regularized if we accept to replace $\varphi(x) = 1$ by a square integrable function which is constant in a finite region only. This well-known procedure would break Poincaré invariance by the introduction of a box in space-time [15].

We prefer an alternative method which preserves invariance but requires an extension of the formalism.

5. ALTERATION OF THE FORMALISM

We are going to redefine divergent quantities as elements of a commutative algebra $\mathbb{A}$ of c-numbers in which complex numbers can be imbedded. At the same time, the space of state vectors will be enlarged as to become a *module* on $\mathbb{A}$ (i.e. linear combinations by elements of $\mathbb{A}$ will be permitted) which implies, of course, that it remains a *vector space* on the field of complex numbers, $\mathbb{C}$.

The underlaying idea is that we should replace the initial Hilbert space by a similar construction of a bigger vector space endowed with a hermitian form which takes its values in $\mathbb{A}$. Indeed such a structure would present almost no formal difference from the traditional one, with the advantage of a tremendous mathematical flexibility.

However we shall not give here a systematic development of this point of view which deserves a separate study [16].

Rather, we are going to introduce as simply as possible the required modifications. Not departing from the Fock formalism, we still postulate that the state vectors are

$$\Phi = (\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots) \tag{5.1}$$

but the nature of the various φ_n is altered by assuming additional dependence on an extra variable ξ. This real and scalar variable is by no means a new degree of freedom. On the contrary it must be understood as mere mathematical parameter, the role of which is to insure consistent calculations.

The generic term of the sequence Φ is thus of the form

$$\varphi_n(x_1, \ldots, x_n, \xi).$$

Let us be more specific by requiring that the dependence on ξ is holomorphic. As the holomorphic functions in ξ (with complex coefficients) form (algebrically speaking) a *field* $\mathbb{A}$, we can consider φ_n not only as a map: $\mathbb{R}^{4n} \times \mathbb{R} \to \mathbb{C}$ but also as a map: $\mathbb{R}^{4n} \to \mathbb{A}$, that is an $\mathbb{A}$-valued function of $x_1, \ldots, x_n$. With this later point of view, the n-body wave function φ_n still can be considered as an element of the n^{th} tensorial power of the one-particle space. Naturally, linear combinations involve coefficients which are arbitrary holomorphic functions in ξ.

We suggest to call *alteration* the above procedure which consists in the introduction of the ξ dependence.

If F is a certain space of functions, we shall note $F^{\#}$ the space obtained by adding the holomorphic dependence on ξ.

From now on dependence on ξ will be implicitly understood as holomorphic.

In order to define precisely the new space of states, let us begin with the one-particle space.

If L is the space made of functions of the form

$$\varphi = f(x) + \lambda \tag{5.2}$$

with $\lambda = \text{const.}$ and $\int_{-\infty}^{\infty} f \, d^4x$ converges [9], we call $L^{\#}$ the space of the functions

$$\varphi(x, \xi) = f(x, \xi) + \lambda(\xi) \tag{5.3}$$

where $\int f(x, \xi) \, d^4x$ converges, and λ depend on ξ.

On L we may define with obvious notations an $\mathbb{A}$-valued sesquilinear map

$$(\varphi, \psi) = \int f^* g \, d^4x + \mu \int f^* \, d^4x + \lambda^* \int g \, d^4x + \lambda^* \mu \xi. \tag{5.4}$$

(Note that (5.4) can also be considered as defining on L an infinite sequence of matrices parametrized by ξ. This point of view will be invoked in our conclusion). In the spirit of a recent work [16] this formula can be extended to $L^{\#}$ which becomes an $\mathbb{A}$-vector space with hermitian form.

Let $L^{\#\otimes n}$ be the n^{th} tensorial power of $L^{\#}$ considered as a space of $\mathbb{A}$-valued functions in the variable x. For instance $L^{\#} \otimes L^{\#}$ is made of functions of the form

$$\varphi_2 = f(x_1, x_2, \xi) + g(x_1, \xi) + h(x_2, \xi) + \mu(\xi) \tag{5.5}$$

where f is integrable from $-\infty$ to $+\infty$ with respect to both x_1 and x_2, g and h are integrable with respect to *respectively* x_1 and x_2.

Naturally we make the convention that the $L^{\#\otimes(0)} = \mathbb{A}$.

Taking $L^{\#\otimes n}$ as n-particle space, we construct the new space of states, say $\Gamma^{\#}$.

As anounced previously, $\Gamma^{\#}$ is a vector space on $\mathbb{C}$, and also on $\mathbb{A}$. It is equipped with a ($\mathbb{A}$-valued) *hermitian form*, thus hermiticity and unitarity have a precise meaning when speaking of operators in $\Gamma^{\#}$.

Extension of the Poincaré group to altered states.

$\Phi = (\ldots, \varphi_n(x_1, x_2 \ldots x_n, \xi), \ldots)$ is straightforward: the usual formulas apply, the scalar variable ξ being ignorable. The same remark holds for the particle number operator and the free hamiltonian generators.

Now we come to the main point of this section:

C can be replaced by a new operator $C^{\#}$ which is defined by similar formulas but is well-defined in $\Gamma^{\#}$.

In this spirit, $C^{\#}$ will be first defined as a map:

$$L^{\#\otimes n} \rightarrow L^{\#\otimes(n-1)}$$

and then extended to $\Gamma^{\#}$. Let us start with the one particle space, and φ as in (5.3). We define

$$C^{\#}\varphi = \int f(x, \xi)\, \mathrm{d}^4 x + \xi\, \lambda(\xi). \tag{5.6}$$

To each φ in $L^{\#}$ we associate by $C^{\#}$ an element of $\mathbb{A}$. When $\lambda \equiv 0$ we recover the definition of $\mathbb{C}$. Note that if f and λ are independent of ξ then φ is in L, but $C^{\#}$ does not map L into $\mathbb{C}$. Therefore the introduction of *altered states* (i.e. states depending on ξ) cannot be avoided. Note that an illegal application of C to a non-trivial element of L (i.e. $\lambda = \text{const.} \neq 0$) would have produced a formula like eq. (5 - 6) with a singularity in place of ξ in the second term.

If Λ is a space-time displacement, it is obvious that

$$C^{\#}\varphi(\Lambda x, \xi) = C^{\#}\varphi. \tag{5.7}$$

We extended $C^{\#}$ to any n-particle state by

$$C^{\#}(\psi_1 \otimes \ldots \psi_n) = C^{\#}\psi_1(\psi_2 \otimes \ldots \psi_n). \tag{5.8}$$

For example we have explicitly in $L^{\#} \otimes L^{\#}$ the formula

$$C^{\#}\varphi_2 = \int f(y, x_1, \xi)\, \mathrm{d}^4 y + \int g(y, \xi)\, \mathrm{d}^4 y + \xi h(x_1, \xi) + \xi \mu(\xi). \tag{5.9}$$

Obviously $C^{\#}\varphi_2$ belongs to $L^{\#}$. Finally $C^{\#}$ is everywhere defined in $\Gamma^{\#}$ by the formula

$$C^{\#}\Phi = (C^{\#}\varphi_1, C^{\#}\varphi_2, \ldots C^{\#}\varphi_{n+1}, \ldots). \tag{5.10}$$

Of course $C^{\#}\Phi$ coincides with $C\Phi$ in the particular case where $\Phi \in H$.

Remark. $C^{\#}\Phi$ does not map into itself the space Γ defined by requiring that $\varphi_n \in L^{\otimes n}$. The intervention of altered states is therefore essential. From (5.7) (5.8) (5.10) we see that $C^{\#}$ commutes with the transformations of the Poincaré group.

Unlike C, the tensorial product operator T can be extended to $\Gamma^{\#}$ without modification.

Indeed the formulas can be used also when the functions φ_n additionally depend on ξ.

Note that $C^{\#}$ and T are conjugate with respect to the hermitian form of $\Gamma^{\#}$.

Finally we can replace $C + T$ by the linear operator

(5.11) $$A = C^{\#} + T$$

which is everywhere defined in $\Gamma^{\#}$. The various powers of A are well-defined, and so is $C^{\#}T$.

Since the action of A coincides with that of $C + T$ is some particular cases, checks made with $C + T$ are sufficient to insure $[A, N] \neq 0$ and $[A, \overline{h}_{n,a_n}]$ not all vanish.

Therefore non-trivial interacting hamiltonians h_{n,a_n} are defined by eq. (3.9) when

(5.12) $$B = \exp ik\, A$$

k being a coupling constant. For the moment eq. (5.12) defines B and B^{-1} as formal series in k, *at least*. In contrast with the situation which would have resulted from eq. (4.2), each term of the series is now well-defined.

We guess that a more elaborated theory of linear operators in $\Gamma^{\#}$ would improve our definition of B.

Similarly, the squared mass operators are defined by (3.10).

Note that in (5.11) A satisfies all the desired requirements.

The invertibility of B is sufficient to insure a one-to-one correspondance between the free mass shell and the interacting mass shell, for each value of the mass. Hence M has a continuous spectrum of generalized eigenvalues which are not only real, but positive.

6. STATES ON THE MASS SHELL

The model is now defined by equations (3.9) - (3.12) where B is the series (5.11). Computation of (3.11) can be carried out at any order in the coupling constant.

As it involves repeated applications of $C^{\#}$ and T, the result can be written explicitly up to a certain number of integrations over space-time, for an arbitrary $\overline{\Phi}$ on the free mass shell.

The following remark is important: We have enlarged the initial formalism as to incorporate systematically altered states. In particular the space $\overline{K}$ of the states which are on the free mass shell has been enlarged too.

For example the one-particle state

$$(0, \xi\, e^{il\cdot x}, 0 \ldots)$$

with $l^2 = m^2$, is an altered solution of (3.6). Its physical interpretation is not obvious in spite of the possiblility that it represents an ultraidealized situation.

If any collision theory is possible at all, the question arises wether it may happen that Φ_{out} be ξ-dependent when Φ_{in} is not.

In the absence of any argument for discarding this possibility we face the problem of interpreting also altered states.

In other words, numerical information (in the conventional sense) should be extracted from altered amplitudes.

In the present formalism, the quantities usually required to have a direct physical meaning (expectation values, transition probabilities, etc.) may be ξ dependent (except if a fortunate cancellation mechanism occurs, which is by no means certain).

The limit of these quantities for $\xi \to +\infty$ will be considered as the relevant numerical answer, in agreement with the heuristic intuition that ξ replaces a divergence.

Let us point out that the calculation of Φ can be explicitly carried out when $\overline{\Phi}$ is an n-body plane wave. Indeed, recalling $C^{\#} 1 = \xi$ ($C^{\#}$ as a map $L \to \mathbb{A}$) and formula (2.10), we see that repeated applications of $C^{\#}$ and T give a result in closed form (at any order in k).

Computation of the physical vacuum is elementary, with the help of the useful formula

$$C^{\#} T^{n} \overline{\Phi}_0 = \xi T^{n-1} \overline{\Phi}_0. \tag{6.1}$$

One finds for instance

$$A\, \overline{\Phi}_0 = T \overline{\Phi}_0 \tag{6.2}$$

$$A^2 \overline{\Phi}_0 = \xi\, \overline{\Phi}_0 + T^2 \overline{\Phi}_0. \tag{6.3}$$

Thus, at the second order

$$\Phi_0 = \left(1 - \frac{k^2}{2}\, \xi\right) \overline{\Phi}_0 + ikT\, \overline{\Phi}_0 - \frac{k^2}{2}\, T^2 \overline{\Phi}_0 \tag{6.4}$$

where $T^n \overline{\Phi}_0$ is the n-body state $(\ldots, 0, 1, 0, \ldots)$.

It was obvious from the very beginning that physical states are generally not normalizable in the sense of the inner product $\langle \Phi, \Psi \rangle$ in H, which is a mathematical structure without direct physical interpretation [17].

Nevertheless, the free-mass-shell space $\overline{K}$ is equipped with a hilbertian structure of its own $\{\Phi, \Psi\}$, (the physical scalar product resulting from properties of the solutions of the Klein-Gordon equation).

Now the physical space which results from taking $\overline{\Phi}$ unaltered in eq. (3.11) can be given a similar structure, just by imposing the definition

$$\{\Phi, \Psi\} = \{\overline{\Phi}, \overline{\Psi}\}.$$

More generally if we wish to incorporate altered states we shall considere an $\mathcal{A}$-vector space with a hermitian form $\{\Phi, \Psi\}$.

7. CONCLUSION AND OUTLOOK

We have succeded in the perturbative construction of the state vectors which represent interacting particles. Expressions in closed from will be availabe if the operator $B = \exp i\, k\, (C^{\#} + T)$ is given a meaning other than perturbative, which leads to investigate the topology of spaces endowed with a hermitian form without being Hilbert spaces.

In view of a simple and comprehensive presentation we have reserved this question for subsequent papers.

Another simplification was taking L as one-particle space, instead of the space of tempered distributions. (L is somehow the minimal space we can afford). But a more general assumption would have drastically complicated the procedure of alteration.

Furthermore we have assumed that the particles are distinguishable and scalar. These properties do not seem to be essential and could be relaxed.

At least one can seek for a model invariant under particle permutations. Besides, future work will pay more attention to the separation of positive energy states.

We do not claim that the alteration procedure developed here is a renormalization method in the full sense.

Indeed we have been substantially helped by the fact that the only divergences in the tentative model of section 4. all arise from the infinite volume of space-time. This is more comfortable than the generic situation one usually meets in realistic Q.F.T.

But some analogy exists with the so-called $(\Phi^4)_2$ theory, which has no ultra violet divergences.

Our altered formalism can be compared with the method utilized therein: Instead of a cut-off labelled by a discrete parameter, we have in fact introduced a (continuous) infinite sequence of metrics, parametrized by ξ, and also a sequence of systems with an interaction which is Poincaré invariant for each value of ξ.

In our scheme P_0 is relieved from the task of generating the dynamics (this role being assigned to the h), which brings out a great flexibility. A question remains: Is it possible to find an invariant model satisfying conditions (a) (b) (c) (d) (I, II) of Section 3 without alteration, but not necessarily in the special form of (3.9) (3.10)?

We have a serious doubut and the most constructive attitude, in the state of our knowledge, is to improve the method of alteration, in view of further constructions.

There remains a lot of work to carry out. Indeed, constructing the space of states is only the first step in the quantization of a system.

The second step (and not the least) is the study of scattering properties. All we have done so far is to pose this problem. The complication involved by the use of an infinitely-many-time formalism may seem hopeless. But it may be noticed that our evolution operator, which depends on a countable infinity of proper-times is less complicated than the one of the Tomonaga-Schwinger formalism, which is labelled by a spacelike surface.

Here, like in the case of Q.F.T., one knows that the tremendous redundance of the formalism (price paid for a top level of generality) can hide a simpler and more intuitive one-parameter picture which is less beautiful but more efficient in practice.

Another way to look at this question can be the search for models in which the number of particles is variable but cannot exceed a fixed bound [4]. For such truncated interactions the many-time formalism will be tractable, and S-matrix calculations possible.

Anyway, promising results obtained in the 2-body [18] and n-body [19] sectors show that a reasonable scattering theory may be founded on a picture which involves many evolution parameters.

Returning to the model presented here, let us stress that if S exists, it has obviously no reason to be the identity.

It will be exciting to make some contact with usual Q.F.T. The difficulty stems from this peculiarity of field theory that its operator formulation is *a priori* restricted to the free-mass-shell space.

Before attacking this problem, it will be useful to construct and study simplified example of this situation.

REFERENCES

[1] For a survey see Relativistic Action at-a-Disctance, Classical and Quantum Aspects (Lecture Notes in Physics), proceedings of the Workshop, Barcelona, Spain 1981, edited by J. LLOSA (Springer, Berlin 1982) Vol. 162.
For an introduction to the *a priori* hamiltonian formalism of Predictive Mechanics, see

Ref. [11].
On the Constraint formulation, see
I.T. TODOROV, J.I.N.R. Report E2-10125, Dubna (1976)
V.V. MOLOTKOV, I.TODOROV, *Gauge dependence of worls-lines and invariance of the S-matrix in Relativistic Classical Mechanics*, Comm. Math. Phys. **79** 111 (1981)
F. ROHRLICH, *Many-body forces and the cluster decomposition*, Phys. Rev. D **23** 1305 (1981)
N. MUKUNDA, E.C.G. SUDARSHAN, *Form of relativistic dynamics with world-lines*, Phys. Rev. D **23**, 2210 (1981).

[2] In so far as classical electrodynamics is concerned, this reformulation is possible, F.J. KENNEDY, *Instantaneous Action-at-a-Distance Formulation of Classical Electrodynamics*, J.M. P. **10** 1349 (1969), L. BEL, A. SALAS, J.M. SANCHEZ, *Approximate solutions of Predictive Relativistic Mechanics for the Electromagnetic interaction*, Phys. Rev. D **7** 1099 (1973)
L. BEL and J. MARTIN, *Approximate solutions of Predictive Relativistic Mechanics for the Electromagnetic interaction II*, Phys. Rev. D **8** 4347 (1973).

[3] P. DROZ-VINCENT, *Second Quantization of directly interacting particles* in *Relativistic Action-at-a-Distance*, (op. cit.) pp. 88 - 101.

[4] F. COESTER, W.N. POLYZOU, *Relativistic Qunatum Mechanics of particles with direct interaction*, Phys. Rev. D **26** 1348 (1982).

[5] In the free case this is a generalized eigenvalue problem in the sense of the theory of rigged Hilbert spaces.
For an introduction to rigged Hilebert spaces see N.N. BOGOLIBOV, A.A. LOGUNOV and I.T. TODOROV, *Introduction to Axiomatic Quantum Field Theory* (Benjamin, Reading Mass., 1975) Chap. I secs. 4 - 6 p. 32 et seq..

[6] P. DROZ-VINCENT, *Une formulation hors couche de masse pour la seconde quantification*, C.R. Ac. Sc. Paris **292** 761 (1981).

[7] The free field of Reference 6 is actually a Generalized free field in the sense of O.W. GREENBERG, *Generalized Free Fields and Models of Local Field Theory*, Ann. of Phys. **16** 158 (1961).

[8] D. KASTLER, *Introduction à l'Electrodynamique quantique*, Dunod, Paris (1961) p. 286.

[9] When φ is in $L^2(\mathbb{R}^4, d^4x)$ its Fourier transformed $\tilde{\varphi}$ is in $L^2(\mathbb{R}^4, d^4 l)$ and $\langle 1, \varphi \rangle$ is proportional to $\tilde{\varphi}(0)$, if $\tilde{\varphi}$ is defined at the origin.

[10] P. DROZ-VINCENT: Ref. 3, p. 98, equation (2 - 13).

[11] For an introduction to Predictive Mechanics in the so-called *a priori* hamiltonian formalism see Ph. DROZ-VINCENT, *Hamiltonian systems of Relativistic Particles*, Reports Math. Phys. **8** 79 (1975)
Two-body relativistic systems, Ann. Inst. H. Poincaré, **27** 407 (1977)
Action-at-a-Distance and relativistic wave equations for spinless quarks, Phys. Rev. D **19** 702 (1979)
N-body Relativistic Systems, Ann. Inst. H. Poincaré **32** 377 (1980) and contributions in Ref. 1.

[12] About the distinction between the physical vacuum and the bare vacuum see, N.N. BOGOLIUBOV, A.A. LOGUNOV and I.T. TODOROV, Op. Cit., Chap 26, p. 561.

[13] R. HAAG, *The framework of Quantum Field Theory*, Nuovo Cimento Suppl., **14** 131 (1959).

[14] See D. KASTLER, Op. Cit. Appendix A, §3, p. 288 - 289.

[15] A.S. WIGHTMAN, *in* Cargèse Lectures in Theoretical Physics High Energy Electromagnetic Interactions and Field Theory. Edited by M. LEVY (New York, Gordon and Breach, 1967), p. 171 - 291.

[16] P. DROZ-VINCENT, *A generalization of Dirac formalism of Quantum Mechanics*, Lett. Nuovo Cimento **40** n. 11, 336 (1984).
Modules with scalar product, Letters in Math. Phys. to be published.

[17] The inner product in H would receive a physical interpretation if the inner product in $L^2(\mathbb{R}^4)$ was to be interpreted, which is still problematic, but has been sometimes proposed: L.P. HORWITZ and F. ROHRLICH, *Constraint Relativistic Quantum Dynamics*, Phys. Rev. D **24** 1528 (1981)
J.R. FANCHI and W.J. WILSON, *Relativistic Many-body systems: Evolution parameter formalism*, Found. Phys. **13** 571 (1983).

[18] P. DROZ-VINCENT, *Two-body relativistic scattering of directly interacting particles*, Phys. Rev. D **29** 687 (1984).

[19] L.P. HORWITZ and F. ROHRLICH, *Scattering in Constraint relativistic quantum dynamics*, Phys. Rev. D **26** 3452 (1982).

Manuscript received: April 10, 1985.